Optimization Practice Worksheet **KEY** **Form D**

Find two positive numbers that satisfy the given requirements.

1. The sum is 243 and the product is a maximum.

$$x + y = 243 \Rightarrow y = 243 - x$$
$$P = x \cdot y = x(243 - x) = 243x - x^2$$
$$P' = 243 - 2x = 0$$
$$243 = 2x$$
$$x = 121.5$$

	MAX	
P'	+ + + + +	0	- - - - - - -	
x	121.5	

$$y = 243 - x = 243 - 121.5 = 121.5$$

$\boxed{121.5,\ 121.5}$

2. The product is 198 and the sum is a minimum.

$$x \cdot y = 198 \Rightarrow y = \frac{198}{x}$$
$$S = x + y = x + \frac{198}{x}$$
$$S' = 1 - \frac{198}{x^2} = 0$$
$$1 = \frac{198}{x^2}$$
$$x^2 = 198$$
$$x = 14.07$$

	MIN	
S'	- - - - - - -	0	+ + + + +	
x	14.07	

$$y = \frac{198}{x} = \frac{198}{14.07} = 14.07$$

$\boxed{14.07,\ 14.07}$

3. The product is 198 and the sum of the first plus four times the second is a minimum.

$$x \cdot y = 198 \Rightarrow y = \frac{198}{x}$$
$$S = x4y = x + \frac{792}{x}$$
$$S' = 1 - \frac{792}{x^2} = 0$$
$$1 = \frac{792}{x^2}$$
$$x^2 = 792$$
$$x = 28.14$$

	MIN	
S'	- - - - - - -	0	+ + + + +
x		28.14	

$$y = \frac{198}{x} = \frac{198}{28.14} = 7.04$$

28.14, 7.04

4. The sum of the first and twice the second is 100 and the product is a maximum.

$$x + 2y = 100 \Rightarrow x = 100 - 2y$$
$$P = x \cdot y = (100 - 2y)(y) = 100y - 2y^2$$
$$P' = 100 - 4y = 0$$
$$4y = 100$$
$$y = 25$$

	MAX	
P'	+ + + + +	0	- - - - - - -
x		25	

$$x = 100 - 2(25) = 100 - 50 = 50$$

25, 50

Optimization Practice Worksheet **KEY** **Form D**

Find the length and width of a rectangle that has the given perimeter and has a maximum area.

5. Perimeter: 125 feet

$$P = 2x + 2y \Rightarrow y = \frac{125 - 2x}{2}$$

$$A = x \cdot y = x\left(\frac{125 - 2x}{2}\right) = \frac{125x - 2x^2}{2}$$

$$A' = \frac{1}{2}(125 - 4x) = 0$$

$$125 = 4x$$

$$x = 31.25$$

		MAX	
A'	+ + + + +	0	- - - - - - -
x		31.25	

$$y = \frac{125 - 2x}{2} = \frac{125 - 2(31.25)}{2} = 31.25$$

31.25 ft by 31.25 ft

6. Perimeter: 200 feet

$$P = 2x + 2y = 200 \Rightarrow y = 100 - x$$

$$A = x \cdot y = x(100 - x) = 100x - x^2$$

$$A' = 100 - 2x = 0$$

$$2x = 100$$

$$x = 50$$

		MAX	
A'	+ + + + +	0	- - - - - - -
x		50	

$$y = 100 - x = 100 - 50 = 50$$

50 ft by 50 ft

Find the length and width of a rectangle that has the given area and has a minimum perimeter.

7. Area: 80 square feet

$$A = x \cdot y = 80 \Rightarrow y = \frac{80}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{80}{x}\right) = 2x + \frac{160}{x}$$

$$P' = 2 - \frac{160}{x^2} = 0$$

$$2 = \frac{160}{x^2}$$

$$2x^2 = 160$$

$$x^2 = 80$$

$$x = 8.94$$

		MIN	
P'	- - - - - - -	0	+ + + + +
x		8.94	

$$y = \frac{80}{x} = \frac{80}{8.94} = 8.94$$

8.94 ft by 8.94 ft

8. Area: 100 square feet

$$A = x \cdot y = 100 \Rightarrow y = \frac{100}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{100}{x}\right) = 2x + \frac{200}{x}$$

$$P' = 2 - \frac{200}{x^2} = 0$$

$$2 = \frac{200}{x^2}$$

$$2x^2 = 200$$

$$x^2 = 100$$

$$x = 10$$

		MIN	
P'	- - - - - - -	0	+ + + + +
x		10	

$$y = \frac{100}{x} = \frac{100}{10} = 10$$

10 ft by 10 ft

Find the point on the graph of the function closest to the given point.

9. $f(x) = \sqrt{x}$, (3, 1)

$$d^2 = (y-1)^2 + (x-3)^2$$
$$d = \sqrt{(y-1)^2 + (x-3)^2}$$
$$d = \sqrt{\left(\sqrt{x}-1\right)^2 + (x-3)^2}$$
$$d' = \tfrac{1}{2}\left(\left(\sqrt{x}-1\right)^2 + (x-3)^2\right)^{-1/2}\left(2\left(\sqrt{x}-1\right)\left(\tfrac{1}{2}x^{-1/2}\right) + 2(x-3)\right) = 0$$
$$\frac{\sqrt{x}-1}{\sqrt{x}} + 2(x-3) = 0$$
$$\sqrt{x}-1 = -2\sqrt{x}(x-3)$$
$$x-\sqrt{x} = -2(x-3)$$
$$-\sqrt{x} = -2x^2 + 6x + x$$
$$-\sqrt{x} = -2x^2 + 7x$$
$$x = 4x^4 - 28x^3 + 49x^2$$
$$0 = 4x^4 - 28x^3 + 49x^2 - x$$
$$x = 2.799$$

		MAX	
d'	+ + + + +	0	- - - - - - -
x		2.8	

$y = \sqrt{x} = \sqrt{2.799} = 1.673$

$\boxed{(2.799,\ 1.673)}$

10. $g(x) = x^3$, (0, 5)

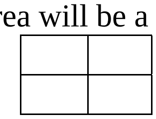
$$d^2 = (y-5)^2 + x^2$$
$$d = \sqrt{(x^3-5)^2 + x^2}$$
$$d' = \tfrac{1}{2}\left((x^3-5)^2 + x^2\right)^{-1/2}\left(\left(2(x^3-5)(3x^2) + 2x\right)\right) = 0$$
$$6x^2(x^3-5) + 2x = 0$$
$$6x^5 - 30x^2 + 2x = 0$$
$$2x(3x^4 - 15x + 1) = 0$$
$$x = 1.687, 0.0667, 0$$

		MAX		MIN	
A'	+ + + + +	0	- - - - - - - - - - - - - -	0	+ + + + +
x		0.07		1.7	

$y = x^3 = 1.687^3 = 1.699$

$\boxed{(1.687,\ 1.699)}$

11. A rancher has 500 feet of fencing with which to enclose four adjacent corrals (see diagram below). What dimensions should be used so that the enclosed area will be a maximum?



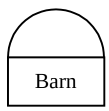
$$P = 6x + 6y = 500 \Rightarrow y = \frac{500-6x}{6}$$
$$A = (2x)(2y) = 4xy = 4x\left(\frac{500-6x}{6}\right) = \frac{200x - 24x^2}{6}$$
$$A' = \tfrac{1}{6}(200 - 48x) = 0$$
$$48x = 2000$$
$$x = 41\tfrac{2}{3}$$

		MAX	
A'	+ + + + +	0	- - - - - - -
x		41.67	

$$y=\frac{500-6x}{6}=\frac{500-6\left(41\tfrac{2}{3}\right)}{6}=\frac{500-250}{6}=\frac{250}{6}=41\tfrac{2}{3}$$

Each corral should be $41\tfrac{2}{3}$ ft by $41\tfrac{2}{3}$ ft.

12. A rancher has 100 feet of fencing with which to enclose a semi-circular area adjacent to a barn (see diagram below). What radius should be used so that the enclosed area will be a maximum?



All of the fence is used to build the semi-circular area, so:

$$\pi r=100$$

$$r=\frac{100}{\pi}=31.83$$

The radius of the area should be 31.83 feet.

13. An open box is to be made from an 8.5 in by 11 in piece of paper by cutting equal squares from each corner and turning up the sides. Find the dimensions of the box of maximum volume.

$$V=x(11-2x)(8.5-2x)=x\left(93.5-39x+4x^2\right)=4x^3-39x^2+93.5x$$

$$V'=12x^2-78x+93.5=0$$

$$x=4.91,1.59$$

		MAX		MIN	
V'	+ + + + +	0	- - - - - - - - - - - - - -	0	+ + + + +
x		1.59		4.91	

$$11-2x=11-2(1.59)=7.83$$

$$8.5-2x=8.5-2(1.59)=5.53$$

The box should be 7.83 in long, 5.53 in wide, and 1.59 in deep.

14. A Norman window is constructed by adjoining a semi-circle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.

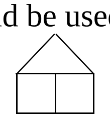
$$P = 2y + 2r + \pi r = 16 \Rightarrow y = 8 - r - \tfrac{\pi}{2} r$$
$$A = 2ry + \tfrac{1}{2}\pi r^2 = 2r\left(8 - 4 - \tfrac{\pi}{2} r\right) + \tfrac{1}{2}\pi r^2 = 16r - 2r^2 - \pi r^2$$
$$A' = 16 - 4r - 2\pi r = 0$$
$$16 = (4 + 2\pi) r$$
$$r = \frac{16}{4 + 2\pi} = 1.556$$

		MAX	
A'	+ + + + +	0	- - - - - - -
r		1.556	

$$y = 8 - r - \tfrac{\pi}{2} r = 8 - 1.556 - \tfrac{\pi}{2}(1.556) = 4$$

Rectangular part: 3.11 ft wide by 4 ft tall, Semi-circular part: radius of 1.56 ft.

15. A rancher has 400 feet of fencing with which to enclose two rectangular corrals adjacent to one triangular corral (see diagram below). What dimensions should be used so that the enclosed area will be a maximum?



$$P = 2y\sqrt{2} + 4x + 3y = 400 \Rightarrow 4x = 400 - 2y\sqrt{2} - 3y \Rightarrow x = 100 - \tfrac{\sqrt{2}}{2} y - \tfrac{3}{4} y$$
$$A = 2xy + \tfrac{1}{2}(2xy) = 2xy + xy = 3xy = 3\left(100 - \tfrac{\sqrt{2}}{2} y - \tfrac{3}{4} y\right)(y) = 300y - \tfrac{3\sqrt{2}}{2} y^2 - \tfrac{9}{4} y^2$$
$$A' = 300 - 3\sqrt{2} y - \tfrac{9}{2} y = 0$$
$$300 = \left(3\sqrt{2} + \tfrac{9}{2}\right) y$$
$$y = \frac{300}{3\sqrt{2} + \tfrac{9}{2}} = 50.725$$

		MAX	
A'	+ + + + +	0	- - - - - - -
y		50.725	

$$x = 100 - \tfrac{\sqrt{2}}{2} y - \tfrac{3}{4} y = 100 - \tfrac{\sqrt{2}}{2}(50.725) - \tfrac{3}{4}(50.725) = 26.088$$

Each rectangular corral: 26.09 ft by 50.73 ft, Triangular corral: base 52.18 ft by height 50.73 ft