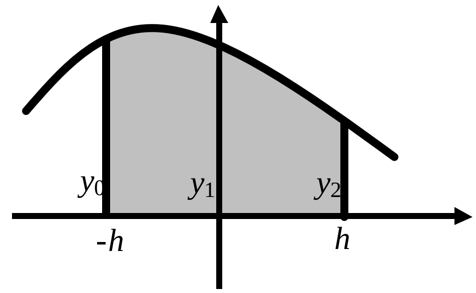
Simpson's Rule



The generic form of a parabola is $y = ax^2 + bx + c$.

The area of the parabolic region from $x = -h$ to $x = h$ is

$$A = \int_{-h}^{h} \left(ax^2 + bx + c\right) dx$$

$$= \left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right]_{-h}^{h} = \left[\frac{a}{3}h^3 + \frac{b}{2}h^2 + ch\right] - \left[\frac{-a}{3}h^3 + \frac{b}{2}h^2 - ch\right]$$

$$= \frac{2a}{3}h^3 + 2ch = \frac{h}{3}\left(2ah^2 + 6c\right) = \frac{h}{3}\left(2ah^2 + 2c + 4c\right)$$

Find y_0, y_1, and y_2 by substituting the *x*-values at those points

$$y_0 = ah^2 - bh + c$$

$$y_1 = c$$

$$y_2 = ah^2 + bh + c$$

Add the y_0 and y_2

$$y_0 + y_2 = \left(ah^2 - bh + c\right) + \left(ah^2 + bh + c\right) = 2ah^2 + 2c$$

Substituting into the area equation above gives

$$A = \frac{h}{3}\left(y_0 + y_2 + 4y_1\right)$$

So, the area of the parabolic region can be found by adding first *y*-value, y_0, four times the middle *y*-value, y_1, and the last *y*-value, y_2, then multiplying that sum by one-third of the interval width, *h*.

Thus, given an odd number of *x*-values, $x_0, x_1, x_2, \ldots, x_{n-2}, x_{n-1}, x_n$, the area of the region can be estimated by

$$A = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \frac{h}{3}\left(y_4 + 4y_5 + y_6\right) + \cdots + \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

Factor out the $\frac{h}{3}$, replace *h* with Δx, and combine like terms to get Simpson's Rule:

$$A = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots 2y_{n-2} + 4y_{n-1} + y_n\right)$$